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Abstract. A new bosonisation procedure based on an exact method of functional integration is applied to the one-dimensional Tomonaga–Luttinger model with forward scattering. This method circumvents the conventional procedure where the fermion field operator is represented by (bosonic) charge and spin density fields. Fictitious auxiliary fields are introduced to deal with the interaction term so that the fermionic degrees of freedom can be integrated out exactly. In addition to the usual description of the system in terms of charge and spin fluctuations, an effective action in terms of the auxiliary fields can be used. The evaluation of the Green function is facilitated if the latter route is taken. Our results for the spinless and spin-$\frac{1}{2}$ cases agree with previous work.

1. Introduction

In 1950, Tomonaga proposed a model of an assembly of spinless and massless Dirac fermions in $1+1$ dimensions. Translated into the language of the non-relativistic degenerate electron gas, this model involves a free-electron dispersion relation linearised about the Fermi level: $\varepsilon(p) = v_f|p|$ (where v_f and p_f are the Fermi velocity and momentum). This allows the Hamiltonian to be diagonalised exactly in terms of the density operators of the fermion field. The original problem is thus transformed into an equivalent theory of non-interacting bosons. This method involves the replacement of the exact commutators of the density fields by their ground-state expectation values. Luttinger (1963) sought to avoid this by separating left- and right-moving fermions into two branches such that $\varepsilon(p) = \pm v_f p$ (for all real p). However, due to the subtleties of unitarily inequivalent Hilbert spaces, the correct treatment of this model was only given by Mattis and Lieb (1965) who revised the commutators of the density operators. They calculated the momentum distribution near the Fermi points and found that the Migdal discontinuity was smoothed out, contrary to the basic hypothesis of the Landau theory of Fermi liquids. It should be noted that this result cannot be obtained from finite-order perturbation theory, illustrating the importance of the concept of bosonisation. Its applicability was extended to the calculation of correlation functions when Mattis (1974) and Luther and Peschel (1974) presented a boson representation for the fermion field operator in terms of the density operators. This had been derived previously by Jordan (1935, 1936a, b, 1937) in the context of a neutrino theory of light. The model was generalised to the spin-$\frac{1}{2}$ case by Luther and Emery (1975) who included back scattering, by Emery *et al* (1976) who included umklapp processes and by Grinstein *et al* (1979) who also considered forward scattering with spin flip. Remarkably, it was found that such models were still exactly solvable

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for given sets of coupling constants. A precise mathematical formulation of this bosonisation procedure was given by Heidenreich *et al* (1980). The comparison with conventional perturbation theory was first made possible by the work of Dzyaloshinsky and Larkin (1973) who were able to identify the cancellation of non-Tomonaga diagrams to all orders. Details of the large volume of work in this field can be found in a review by Sólyom (1979).

The boson representation was further developed when Haldane (1979) pointed out that the zero-mode terms associated with the particle-number operators should be included. This allows charge and current excitations. Moreover, Haldane (1981) extended the theory to treat the deviation of the fermion spectrum from linearity as corrections to the linearised spinless theory. In the boson representation, this 'Luttinger liquid theory' gives rise to a non-linear coupling among the original elementary excitations. It was shown that much of the low-energy structure of the linearised model was preserved. It was also suggested that a large class of one-dimensional quantum fluids could be described in this way. An example is the Heisenberg–Ising spin chain for which, as shown independently by Fowler (1980), even the linearised model gives a good estimate of the spin-wave phase velocity.

The concept of bosonisation is also well known in relativistic field theory. In particular, it was introduced by Klaiber (1968) for the study of the massless Thirring model which is analogous to the Tomonaga–Luttinger model considered here. Coleman (1975) was able to establish the equivalence between the massive Thirring and sine-Gordon models by direct comparison of their respective perturbation series. The fermions were mapped onto the soliton sector of the sine-Gordon model (Mandelstam 1975). The operator fit of Lowenstein and Swieca (1971) is the central tool for the bosonisation of these Abelian theories. Non-Abelian bosonisation in the operator formalism was initiated by the work of Polyakov and Wiegmann (1983) and Witten (1984). For instance, the chiral-invariant Gross–Neveu model (Gross and Neveu 1974) was extensively studied. A discussion of the developments in this field is beyond the scope of this paper.

Bosonisation in the operator formalism, albeit powerful and efficient, may be quite baffling to the uninitiated. It is known that field-theoretical quantities such as the Green and vertex functions can be expressed directly as functional averages, i.e. by integration over infinite-dimensional function spaces. In condensed matter physics, most of the problems treated in this manner involve approximate calculations of these integrals by the method of steepest descent or the use of background fields. An introduction can be found in the excellent treatise by Popov (1987). The object of this paper is simply to forego the operator methods and derive the bosonisation of the Tomonaga–Luttinger model by an exact method of functional integration. We will not consider any large-momentum scattering processes.

Although this approach has only been touched upon in the literature of the one-dimensional electron gas (Fogedby 1976), it has been employed extensively in relativistic field theory. The original motivation for such work was to provide a procedure for non-Abelian bosonisation (Roskies and Schaposnik 1981, Gamboa Saraví *et al* 1981, 1984, Furuya *et al* 1982). In this context, the chiral non-invariance (Fujikawa 1979, 1980) of the integration measure of the Dirac fields is central to the bosonisation. Naón (1985) and Eides (1985) provide a recent introduction to this area. We present a parallel approach for the Tomonaga–Luttinger model and arrive at the equivalent boson theory without the explicit use of the chiral transformation or the non-trivial Fujikawa procedure needed to evaluate the relevant Jacobian.

This paper is organised as follows. In § 2, the spinless Tomonaga-Luttinger model is defined. Auxiliary fields have to be introduced to express the interaction in terms of the charge density fields. A quadratic action in terms of the charge density field is found by integrating out all other degrees of freedom and the partition function in the canonical ensemble is calculated exactly. It will be seen that this is possible because the interaction of the fermions with the auxiliary fields can be absorbed by a simple gauge transformation. From the free energy, we can then determine the central charge of the problem as being unity, using a result in conformal field theory (Blöte *et al* 1986, Affleck 1986). The system can also be described by an effective action in terms of the auxiliary fields. The single-particle Green function, the momentum distribution and the single-particle density of states are then calculated using, for simplicity, the cutoff procedure of Luther and Peschel (1974). Our results agree with Luther and Peschel (1974), Mattis and Lieb (1965) and Suzumura (1980). The spin-$\frac{1}{2}$ generalisation can be found in § 3, where the partition function is expressed in terms of uncoupled spin density and charge density fluctuations. The Green function, first determined by Dzyaloshinsky and Larkin (1973) by diagrammatic techniques, is reproduced here in the functional framework. Section 4 concludes with a discussion of projects for the future.

2. The spinless Tomonaga–Luttinger model

2.1. The model

In the Tomonaga-Luttinger model, there are two branches of spinless and massless fermions on a line of length L. The fermions in branch 1, described by the field $\psi_1(x)$, move in the positive x direction with velocity v_f, while branch-2 fermions $\psi_2(x)$ move in the opposite direction with velocity $-v_f$. In other words, we have linearised the dispersion relation for the case of free fermions. We allow each branch to have both positive and negative momenta so that we have a sea of negative-energy states which should be filled. A cutoff procedure is needed to prevent these states from contributing to the results of the calculations in this model (see § 2.3). The Hamiltonian is $H = H_0 + H_1$ where

$$\begin{aligned} H_0 &= v_f \int_0^L \mathrm{d}x\, \bar{\psi}(x)(\sigma_3 p - p_f)\psi(x) \\ H_1 &= \int_0^L \mathrm{d}x \int_0^L \mathrm{d}y \sum_{ab} \bar{\psi}_a(x)\psi_a(x) V_{ab}(x-y)\bar{\psi}_b(y)\psi_b(y) \end{aligned} \tag{1}$$

where $p = -\mathrm{i}\partial_x$, $\psi = \binom{\psi_1}{\psi_2}$, $\bar{\psi} = (\bar{\psi}_1, \bar{\psi}_2)$ and $a, b = 1, 2$. We have shifted the origin of the energy scale to $v_f p_f$ to ensure that all states below the Fermi level are occupied in the non-interacting ground state. For a realistic H_1, we choose V_{ab} to be the matrix

$$\begin{pmatrix} v_1 & v_2 \\ v_2 & v_1 \end{pmatrix}.$$

This interaction H_1 represents the forward scattering of the fermions. We have neglected the back scattering and umklapp processes where the momentum changes are large enough ($\sim 2p_f$ and $\sim 4p_f$ respectively) to transfer fermions from one branch to the other. In the conventional Tomonaga-Luttinger model, the interaction is chosen to be local, i.e. $\sim\delta(x-y)$. However, in the next section, we will leave it in a non-local

form for the sake of generality. Our Hamiltonian reduces to that of Luttinger when $v_1 = 0$ and that of Mattis and Lieb when $v_1 = v_2$.

2.2. The partition function

In the imaginary-time formalism, the action for the Hamiltonian of (1) is

$$S = \int_0^\beta d\tau \int_0^L dx\, \bar{\psi}(x,\tau)\partial_\tau \psi(x,\tau) + \int_0^\beta d\tau\, H(\tau) \tag{2}$$

where $H(\tau)$ is the Hamiltonian in the Heisenberg picture:

$$H(\tau) = v_f \int_0^L dx\, \bar{\psi}(x,\tau)(\sigma_3 p - p_f)\psi(x,\tau) + \int_0^L dx \int_0^L dy \sum_{ab} \bar{\psi}_a(x,\tau)\psi_a(x,\tau) V_{ab}(x-y)\bar{\psi}_b(y,\tau)\psi_b(y,\tau)$$

and $\bar{\psi}_a$, ψ_a are independent Grassmann fields. The regularised partition function is given by the formal ratio:

$$Z/Z_0 = \int d[\bar{\psi}]\, d[\psi] \exp(-S[\bar{\psi},\psi]) \left(\int d[\bar{\psi}]\, d[\psi] \exp(-S_0[\bar{\psi},\psi]) \right)^{-1} \tag{3}$$

where $d[\bar{\psi}]\, d[\psi]$ denotes the appropriate integration measure for the anticommuting fermion fields and S_0 is the action which corresponds to the non-interacting case where $V_{ab} = 0$. In calculating the partition function, we integrate over all configurations of the Grassmann fields satisfying antiperiodic boundary conditions in the imaginary time τ (i.e. $\psi(x,0) = -\psi(x,\beta)$ and $\bar{\psi}(x,0) = -\bar{\psi}(x,\beta)$ for $0 < x < L$) and periodic boundary conditions in x. However, the integration of $\exp(-S)$ cannot be performed directly because of the four-fermion term. This difficulty can be avoided by observing that this term can also be written as $H_1 = \int dx \int dy\, \Sigma_{ab}\, \rho_b V_{ab} \rho_b$ where $\rho_a = \bar{\psi}_a \psi_a$ are the density fields of the fermions in the two branches. Thus

$$\exp\left(-\int\int \bar{\psi}_a \psi_a V_{ab} \bar{\psi}_b \psi_b \right) = \int \prod_a d[\rho_a] \exp\left(-\sum_{ab} \int\int \rho_a V_{ab} \rho_b \right) \prod_a \delta[\rho_a - \bar{\psi}_a \psi_a]. \tag{4}$$

The delta functional above behaves like an ordinary delta function and can be represented by a functional integral: $\delta[X] = N \int d[\phi] \exp(i \int \phi X)$ where ϕ is a real commuting field and N is an irrelevant normalisation constant. Using this identity, we can write (3) as

$$\begin{aligned} \frac{Z}{Z_0} = \int \prod_a d[\phi_a]\, d[\rho_a] \exp\left(-\sum_{ab} \int \rho_a V_{ab} \rho_b + i \sum_a \int \rho_a \phi_a \right) \int d[\bar{\psi}]\, d[\psi] \\ \times \exp\left(-\int\int \bar{\psi}(D + i\Phi)\psi \right) \\ \times \left[\int \prod_a d[\phi_a]\, d[\rho_a] \exp\left(i \sum_a \int \rho_a \phi_a \right) \right. \\ \left. \times \int d[\bar{\psi}]\, d[\psi] \exp\left(-\int\int \bar{\psi}(D + i\Phi)\psi \right) \right]^{-1} \end{aligned} \tag{5}$$

where

$$D = \partial_\tau - i\sigma_3 \partial_x - p_f = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix} \qquad \Phi = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix}.$$

Now, we integrate out $\bar{\psi}$ and ψ to obtain the partition function in terms of ρ_a and the auxiliary fields ϕ_a. The integral over the fermion fields which appears in both the numerator and the denominator of the expression in (5) can be rewritten as a product of two Matthews-Salam determinants (Matthews and Salam 1955):

$$\begin{aligned} I_f &= \int \mathrm{d}[\bar{\psi}]\,\mathrm{d}[\psi]\exp\left(-\int\int \bar{\psi}(D+\mathrm{i}\Phi)\psi\right) \\ &= \int \mathrm{d}[\bar{\psi}_1]\,\mathrm{d}[\psi_1]\exp\left(-\int\int \bar{\psi}_1(D_1+\mathrm{i}\phi_1)\psi_1\right)\int \mathrm{d}[\bar{\psi}_2]\,\mathrm{d}[\psi_2] \\ &\quad\times\exp\left(-\int\int \bar{\psi}_2(D_2+\mathrm{i}\phi_2)\psi_2\right) \\ &= \det[D_1+\mathrm{i}\phi_1]\det[D_2+\mathrm{i}\phi_2]. \end{aligned} \tag{6}$$

Using $\det A = \exp(\mathrm{Tr}\ln A)$ and $\ln(A+B) = \ln A + B\int_0^1 \mathrm{d}\lambda\,(A+\lambda B)^{-1}$, we write

$$\det(D_a+\mathrm{i}\phi_a) = \det D \times\exp\left(-\mathrm{i}\int_0^L \mathrm{d}x\int_0^\beta \mathrm{d}\tau\int_0^1 \mathrm{d}\lambda\ \phi_a(x,\tau)G_a(x,\tau;x',\tau',[\phi_a])\Big|_{x'\to x,\tau'\to\tau}\right) \tag{7}$$

where $G_a(x,\tau;x',\tau',[\phi_a])$ is the Green function of $D_a+\mathrm{i}\phi_a$ satisfying

$$[D_a+\mathrm{i}\phi_a(x,\tau)]G_a(x,\tau;x',\tau',[\phi_a]) = -\delta(x-x')\delta(\tau-\tau'). \tag{8}$$

Equation (8) is a first-order linear differential equation for $G_a[\phi_a]$. Therefore, by introducing a local U(1) gauge transformation to the fermion fields, we can reduce $G_a[\phi]$ to the non-interacting Green function G_a^0. In other words, $G_a[\phi]$ and G_a^0 differ from each other only by a phase change and we have the factorisation ansatz (Schwinger 1962)

$$G_a(x,\tau;x',\tau',[\phi_a]) = G_a^0(x-x',\tau-\tau')\exp(f_a(x,\tau,[\phi_a])-f_a(x',\tau',[\phi_a])) \tag{9}$$

where $D_aG_a^0(x,\tau) = -\delta(x)\delta(\tau)$. By observing that D_1D_2 is the two-dimensional Laplacian whose corresponding Green function is $(-1/2\pi)\ln(x^2+\tau^2)^{1/2}$, we can easily see that

$$G_a^0(x,\tau) = (1/2\pi)\exp(\pm\mathrm{i}p_f x)/(\pm\mathrm{i}x-\tau). \tag{10}$$

Here and henceforth, the upper (lower) sign corresponds to $a=1$ ($a=2$). From (8) and (9), we find

$$(D_a+p_f)f_a(x,\tau,[\phi_a]) = -\mathrm{i}\phi_a(x,\tau). \tag{11}$$

Note that ϕ has to be a boson field, periodic in x and τ, so that the Matsubara Green functions satisfy the appropriate fermionic boundary conditions. In the Fourier representation, (11) gives

$$f_a(x,\tau,[\phi_a]) = (1/\beta L)\sum_{p\omega} f_a(p,\omega,[\phi_a])\exp[\mathrm{i}(px-\omega\tau)]$$

where

$$f_a(p,\omega,[\phi_a]) = \mathrm{i}\phi(p,\omega)/(\mathrm{i}\omega\mp p). \tag{12}$$

In the imaginary-time formalism, ω denotes $\omega_n = 2\pi n/\beta$ where $\beta = 1/T$ and n is an integer. From now on, we shall consider only the infinite-size and zero-temperature limit where $(\beta L)^{-1}\Sigma_{p\omega} \to \int_{-\infty}^{\infty} dp/(2\pi) \int_{-\infty}^{\infty} d\omega/(2\pi)$ and we shall use the sum and the integral interchangeably.

It is important to realise that a quadratic dispersion relation for the fermions would give rise to a second-order differential equation for $G_a[\phi]$ and the ansatz would then fail to give a closed equation for f_a as in (12).

Equation (10) shows that $G_a^0(x, 0) \sim \pm 1/2\pi i x$ at short distances. Since G_a^0 and $G_a[\phi]$ differ only by a phase change, this divergence also appears in $G_a[\phi]$ and has to be regularised for (7) to be well defined. Among the existing procedures, we find that the point-splitting method (Schwinger 1962) is best suited to our purpose. This defines the coincident limit of $G_a[\phi]$ to be symmetrical in space with τ' set equal to τ:

$$\begin{aligned} G_a(x, \tau; x', \tau', [\phi])|_{x'\to x, \tau'\to\tau} &:= \tfrac{1}{2}(\lim_{\eta\to 0+} + \lim_{\eta\to 0-}) G_a(x, \tau; x+\eta, \tau, [\phi]) \\ &\sim \pm(1/4\pi i\eta)[(1+\eta\partial_x f_a) - (1-\eta\partial_x f_a)] \qquad (\text{as } \eta \to 0) \\ &= \pm(i/2\pi\beta L)\sum_{p\omega} p\phi(p, \omega)(i\omega \mp p)^{-1} \exp[i(px - \omega\tau)]. \end{aligned} \tag{13}$$

Therefore, from (6), (7) and (13),

$$I_f = (1/4\pi\beta L) \det D \exp\left(-\sum_{p\omega} [p|\phi_1(p, \omega)|^2/(i\omega - p) - p|\phi_2(p, \omega)|^2/(i\omega + p)] \right) \tag{14}$$

where we have used $\phi_a(p, \omega) = \phi_a^*(-p, -\omega)$. Using (5), the partition function becomes

$$\begin{aligned} \frac{Z}{Z_0} = \int \prod_a d[\phi_a]\, d[\rho_a] \exp(-W[\phi_a, \rho_a, V_{ab}]) \\ \times \left(\int \prod_a d[\phi_a]\, d[\rho_a] \exp(-W[\phi_a, \rho_a, 0]) \right)^{-1} \end{aligned} \tag{15}$$

where

$$W[\phi, \rho, V] = (1/\beta L)\sum_{p\omega} \left(\sum_a A_a|\phi_a|^2 + \sum_{ab} \rho_a V_{ab} \rho_b - i\sum_a \rho_a \phi_a \right)$$

is the effective action for the system in terms of ϕ_a and ρ_a with

$$\begin{aligned} &A_1 := -(1/4\pi)p/(i\omega - p) \qquad A_2 := A_1^* = (1/4\pi)p/(i\omega + p) \\ &A := A_1 + A_2 = 8\pi A_1 A_2 = (1/2\pi)p^2/(p^2 + \omega^2). \end{aligned} \tag{16}$$

The effective action $W[\phi, \rho, V]$ is a quadratic functional, thus enabling us to perform the functional integration exactly. We integrate out ϕ_a first so that we are left with an effective action in ρ_a:

$$Z/Z_0 = \int d[\rho] \exp\left(-\int \rho^* \Omega \rho \right) \left(\int d[\rho] \exp\left(-\int \rho^* \Omega_0 \rho \right) \right)^{-1} \tag{17}$$

where

$$\rho = \begin{pmatrix} \rho_1(p, \omega) \\ \rho_2(p, \omega) \end{pmatrix} \qquad \rho^* = (\rho_1^*(p, \omega), \rho_2^*(p, \omega))$$

$$\Omega = \begin{pmatrix} (4A_1)^{-1} + v_1 & v_2 \\ v_2 & (4A_2)^{-1} + v_1 \end{pmatrix}_{p,\omega}.$$

$\Omega_0 = \Omega(v_1 = v_2 = 0)$ and $v_a(p)$ is the Fourier transform of $v_a(x)$. Therefore, we have arrived at a system of non-interacting bosons ρ_a which are the elementary excitations (plasmons) of the original system of interacting fermions.

From (17), we can read off the density response function (in the imaginary-time formulation) as

$$N_{ab}(p, \omega) = \tfrac{1}{2}[\Omega^{-1}(p, \omega)]_{ab}$$

where $N_{ab}(p, \omega)$ is the Fourier transform of $\langle T_\tau \rho_a(x, \tau)\rho_b(0, 0)\rangle$. Performing the analytic continuation to real frequencies, we obtain the plasmon spectrum from the condition $\det \Omega(p, -i\varepsilon) = 0$:

$$\varepsilon(p)^2 = p^2[(1 + v_1(p)/\pi)^2 - (v_2(p)/\pi)^2]. \tag{18}$$

Therefore, the total response function in the imaginary-time formulation can be written as

$$N(p, \omega) = \sum_{ab} N_{ab}(p, \omega) = -(p^2/\pi)[1 + (v_1 - v_2)/\pi]/(\varepsilon(p)^2 + \omega^2). \tag{19}$$

Now, we proceed to evaluate the partition function by integrating out the density fields ρ_a. We obtain

$$Z/Z_0 = \prod_{p\omega} [1 + B(p)/(p^2 + \omega^2)]^{-1/2}$$

$$= \exp\left(-\tfrac{1}{2} \sum_{pn} \ln\{1 + B(p)/[p^2 + (2n\pi/\beta)^2]\} \right)$$

where $B(p) := p^2\{2v_1(p)/\pi + [(v_1(p)/\pi)^2 - (v_2(p)/\pi)^2]\}$. But

$$\sum_n \ln[1 + B(p)/(p^2 + \omega_n^2)]$$

$$= \int_0^1 d\lambda \sum_n B(p)/(\lambda B(p) + p^2 + \omega_n^2)$$

$$= (\beta B(p)/2) \int_0^1 d\lambda (p^2 + \lambda B(p))^{-1/2} \coth[(\beta/2)(\lambda B(p) + p^2)^{1/2}]$$

provided that $B(p)/p^2 + 1 > 0$ for stability. Otherwise the free energy does not exist. The free energy is

$$F(T) = F_0(T) + \sum_{p>0} (\varepsilon(p) - p) + \sum_{p>0} 2T \ln[1 - \exp(-\varepsilon(p)/T)]/[1 - \exp(-p/T)] \tag{20}$$

where we have chosen the units such that v_f, $\hbar$ and the Boltzmann constant are unity. The first term in (20) is the free energy of the non-interacting case while the second is the (infinite) vacuum renormalisation energy. The latter does not contribute to the thermodynamics and, dropping it, we may write $F(T) = 2T \Sigma_{p>0} \ln[1 - \exp(-\varepsilon(p)/T)]$, as is appropriate for a system whose doubly degenerate elementary excitations are bosonic with the eigenvalue spectrum $\varepsilon(p)$. By setting $v_1 = 0$ and $v_2 = v$, we have the Luttinger spectrum $\varepsilon_L(p) = |p|[1 - (v/\pi)^2]^{-1/2}$ and, by setting $v_1 = v_2 = v$, the Mattis-Lieb solution $\varepsilon_{ML}(p) = |p|(1 + 2v/\pi)^{-1/2}$. In the case of a long-range Coulomb interaction, e.g. $v_1 = v_2 = \lambda/p^2$, the plasmons acquire a dynamical mass: $\varepsilon(p) = (p + m^2)^{1/2}$ with $m^2 = 2\lambda/\pi$ (Mattis and Lieb 1965). As $T \to 0$, $F(T) \to -(Tm/\pi)K_1(m/T) \to -(m/2\pi)^{1/2} T^{3/2} \exp(-m/T)$, which is different from the behaviour of a Fermi liquid. This reminds us of the similar situation in $(QED)_2$ where

the photon is found to acquire a mass through the dynamics (Schwinger 1962). We may also consider a local interaction: $v_1(x) = v_1\delta(x)$ and $v_2(x) = v_2\delta(x)$. We find that

$$F(T) = 2T\int \mathrm{d}p/(2\pi)\ln[1-\exp(-pv_f^*/T)]$$

where $v_f^* = v_f[(1+v_1/\pi)^2-(v_2/\pi)^2]^{1/2}$ is interpreted as the renormalised Fermi velocity.

2.3. The single-particle Green function at $T = 0$

We first calculate the Matsubara Green function for the branch-1 electrons:

$$G_1(x,\tau;x',\tau') = -\langle T_\tau[\psi_1(x,\tau)\bar{\psi}_1(x',\tau')]\rangle = -\langle\psi_1(x,\tau)\bar{\psi}_1(x',\tau')\rangle$$

where the second equality is defined in the functional integral form. A Wick rotation then gives the zero-temperature Green function:

$$\mathrm{i}\mathcal{G}_1(x,t;x',t') = \langle T_t[\psi_1(x,t)\bar{\psi}_1(x',t')]\rangle.$$

In the previous subsection, we have already obtained the electron Green function $G_1(x,\tau;x',\tau'[\phi_1])$ for a given configuration of the auxiliary field ϕ_1. Therefore, in order to obtain the full Green function $G_1(x,\tau;x',\tau')$, we need an effective action $\Lambda[\phi_a]$ in terms of ϕ_a only. In order to obtain this explicitly, we integrate out first the density field ρ_a from the functional-average definition of $G_1(x,\tau;x',\tau')$ above. We then arrive at

$$G_1(x,\tau;x',\tau') = \int \mathrm{d}[\bar{\psi}]\,\mathrm{d}[\psi]\,\mathrm{d}[\phi]\,\bar{\psi}_1(x,t)\psi_1(x',t')\exp(-\Gamma[\phi,\psi])$$
$$\times\left(\int \mathrm{d}[\bar{\psi}]\,\mathrm{d}[\psi]\,\mathrm{d}[\phi]\exp(-\Gamma[\phi,\psi])\right)^{-1}$$

where

$$\Gamma[\phi,\psi] = \int\left(\sum_a \bar{\psi}_a(D_a+\mathrm{i}\phi_a)\psi_a + \sum_{ab}\phi_a V_{ab}^{-1}\phi_b/4\right). \tag{21}$$

We now integrate over the fermion fields, getting the Matthews-Salam determinant of the operator $D_1+\mathrm{i}\phi_1$ in the numerator of (21). This has already been calculated. So

$$G_1(x,\tau;x',\tau') = \int\prod_a \mathrm{d}[\phi_a]\,G_1(x,\tau;x',\tau'[\phi_1])\exp(-\Lambda)\left(\int\prod_a \mathrm{d}[\phi_a]\exp(-\Lambda)\right)^{-1}$$
$$= G_1^0(x,\tau;x',\tau')\int\prod_a \mathrm{d}[\phi_a]\exp\{f_1(x,\tau,[\phi_1])-f_1(x',\tau',[\phi_1])-\Lambda[\phi_a]\}$$
$$\times\left(\int\prod_a \mathrm{d}[\phi_a]\exp\{-\Lambda[\phi_a]\}\right)^{-1} \tag{22}$$

where $\Lambda[\phi_a] := \int\Sigma_{ab}\,\phi_a V_{ab}^{-1}\phi_b/4 + \int\Sigma_a A_a|\phi_a|^2$ is the effective action we require. From (12), we have

$$f_a(x,\tau,[\phi_1])-f_a(x',\tau',[\phi_1]) = (\mathrm{i}/\beta L)\sum_{p\omega}\phi_a(p,\omega)J_a(-p,-\omega;x,\tau;x',\tau')$$

where

$$J_a(-p,-\omega;x,\tau;x',\tau')$$
$$:= -\exp[\mathrm{i}(px-\omega\tau)]-\exp[\mathrm{i}(px'-\omega\tau')]\}/(\pm p-\mathrm{i}\omega).$$

Therefore, the integral in (22) is

$$\int\prod_a \mathrm{d}[\phi_a]\exp\left(-\Lambda[\phi_a]+\mathrm{i}\sum_a\rho_a\phi_a+\mathrm{i}\phi_1 J_1\right)\left[\int\prod_a \mathrm{d}[\phi_a]\exp\left(-\Lambda[\phi_a]+\mathrm{i}\sum_a\rho_a\phi_a\right)\right]^{-1}$$

so that

$$G_1(x, \tau; x', \tau') = G_1^0(x, \tau; x', \tau') \exp[-Q(x-x', \tau-\tau')]$$

where $Q(x, \tau)$ is given by

$$-(2\pi/\beta L) \sum_{p\omega} p^{-1} e^{-\alpha|p|}[1-\cos(px-\omega\tau)][(p+i\omega)/(p^2+\omega^2)-(fp+i\omega)/(g^2p^2+\omega^2)] \tag{23}$$

with $f(p) = 1 + v_1(p)/\pi$ and $g(p)^2 = (1+v_1(p)/\pi)^2 - (v_2(p)/\pi)^2$. The convergence factor $e^{-\alpha|p|}$ in momentum integration is introduced as an *ad hoc* ultraviolet cutoff procedure (see below). We shall henceforth consider only local interactions so that f and g are independent of p.

The expression for Q involves elementary but tedious integrations. It can be found in appendix 1 where the analytic continuation to real time $Q(x, \tau) \to Q(x, t)$ has also been performed. Using

$$\mathcal{G}_1^0(x, t) = (2\pi)^{-1} \exp(ip_f x)[(x-t+i\alpha)^{-1} + \text{CC}] \tag{24}$$

we get the final expression for $\mathcal{G}_1(x, t)$:

$$\mathcal{G}_1(x, t) = (2\pi)^{-1} \exp(ip_f x)\{\Theta(t)(x-gt+i\alpha)^{-1}[(x-gt+i\alpha)(x+gt-i\alpha)/\alpha^2]^{-\gamma} + \Theta(-t)[\text{CC}]\} \tag{25}$$

where $\gamma = [(f/g)-1]/2$.

We note that the exponent γ depends on the strength of the interaction. This is a general feature of theories with a conformal charge of unity. The conformal charge, c, can be inferred from the low-temperature expression for the free energy: $F(T) \sim -(\pi/6)cT^2/v_f^*$ (Blöte *et al* 1986, Affleck 1986).

The retarded Green function G_R of Luther and Peschel can be determined from the relation: $\mathcal{G}_{R1}(x, t) = i\Theta(t)[\mathcal{G}_1(x, t) - \mathcal{G}_1(x, -t)]$. Due to the fact that we have chosen the same cutoff for the free-electron Green function and the space-time-dependent decay phase factor Q, we are also able to obtain both the momentum distribution and the single-particle density of states in a closed form.

We have taken the simplest cutoff procedure with only one cutoff parameter. However, in principle, there should be two distinct cutoff parameters in the fermion model, α^{-1} and r^{-1}, for the bandwidth and the momentum transfer respectively. Due to the artificial Fermi sea of negative-energy states, the former is necessary for the regularisation of the non-interacting model. In this interpretation, the convergence factor above amounts to altering the density of states in momentum space. On the other hand, r is associated with the range of the pair potential $V(x-y)$. By neglecting this distinction, we follow the original procedure of Luther and Peschel (1974). Our expression for the Green function will turn out to be the same as their expression. It can be shown that this procedure is consistent with diagrammatic summation in the fermion model when only forward scattering is considered (see, for example, Sólyom 1979). However, for models with large momentum transfer, this treatment is inadequate, producing unphysical results in the limit $\alpha \to 0$. The inclusion of two cutoff parameters in the fermion model was studied by Chui *et al* (1974), Grest *et al* (1976) and Sólyom (1975). Theumann (1977) pointed out that even this procedure was inconsistent. For example, the zero limits of α and r have to be taken in different orders when considering different quantities (see also Fowler 1980). More recently, α has been given a more

natural interpretation in terms of the Jordan point-splitting regularisation where the product $\bar{\psi}_a(x,\tau)\psi_a(x,\tau)$ is defined as $\bar{\psi}_a(x\pm i\alpha/2,\tau)\psi_a(x\mp i\alpha/2,\tau)$ in the limit $\alpha\to 0$. This was implicit in the work of Haldane (1979, 1981) and was discussed in detail later by Apostol (1983) and Apostol *et al* (1987). This is believed to resolve the inconsistencies found in the previous calculations.

2.4. The momentum distribution at $T=0$

The momentum distribution of branch-1 fermions is defined, in the limit of infinite size, as

$$n_1(p) = -i\int_{-\infty}^{\infty} dx\,\exp(-ipx)\lim_{\tau\to 0-} G_1(x,\tau)$$
$$= (2\pi i)^{-1}\int_{-\infty}^{\infty} dx\,\exp[i\alpha x(p_f-p)](x-i)^{-\gamma-1}(x+i)^{-\gamma}. \tag{26}$$

The integral is evaluated in appendix 2. We obtain

$$n_1(p) = |(p-p_f)\alpha|^{2\gamma}\exp[-|(p-p_f)\alpha|][\Theta((p-p_f)\alpha)\Psi(\gamma+1,2\gamma+1;2(p-p_f)\alpha)/\Gamma(\gamma)$$
$$+\Theta((p_f-p)\alpha)\Psi(\gamma,2\gamma+1;2(p_f-p)\alpha)/\Gamma(\gamma+1)] \tag{27}$$

where $\Psi(\alpha,\beta;z)$ is a degenerate hypergeometric function and is related to ${}_1F_1(\alpha,\beta;z)$ by

$$\Psi(\alpha,\beta;z) = {}_1F_1(\alpha,\beta;z)\Gamma(1-\beta)/\Gamma(\alpha-\beta+1)$$
$$+{}_1F_1(\alpha-\beta+1,2-\beta;z)z^{1-\beta}\Gamma(\beta-1)/\Gamma(\alpha).$$

For small z, ${}_1F_1(\alpha,\beta;z) = 1+(\alpha/\beta)z+[\alpha(\alpha+1)/\beta(\beta+1)]z^2/2+O(z^3)$. This power series allows us to expand about $p=p_f$ to get

$$n_1(p_f) = \Gamma(\gamma+\tfrac{1}{2})/2\sqrt{\pi}\Gamma(\gamma+1)$$

and

$$n_1(p)-n_1(p_f) \sim -\tfrac{1}{2}\mathrm{sgn}(p-p_f)|(p-p_f)\alpha|^{2\gamma}/\cos(\pi\gamma) \qquad \text{for } 2\gamma<1$$
$$\sim |(p-p_f)\alpha| \qquad \text{for } 2\gamma>1. \tag{28}$$

Therefore the jump discontinuity at the Fermi surface is obliterated for sufficiently strong interactions—a fact known from the original work of Luttinger, Mattis and Lieb. This means that the Landau Fermi liquid theory is invalid for the Tomonaga-Luttinger model because quasiparticles with long lifetimes can no longer be defined.

2.5. The density of states at $T=0$

The density of states can be obtained from the retarded Green function using the following formula (Suzumura 1980):

$$D(\omega) = -2\int_0^{\infty} dt\,\cos(\omega t)\,\mathrm{Im}\,\mathcal{G}_{R1}(x,t)\Big|_{x=0} \qquad (\omega>0). \tag{29}$$

From the expression for $\mathcal{G}_1(x,t)$

$$\mathcal{G}_{R1}(x,t) = i\Theta(t)\pi^{-1}\exp(ip_f x)\,\mathrm{Im}\{(x-gt+i\alpha)^{-1}[(x/\alpha)^2+(1+igt/\alpha)^2]^{-\gamma}\}.$$

Therefore

$$D(\omega) = -2\pi^{-1} \int_0^\infty \mathrm{d}t \cos(\omega t)\, \mathrm{Im}[(-gt+\mathrm{i}\alpha)^{-1}(1+\mathrm{i}gt/\alpha)^{-\gamma}]$$
$$= (\omega\alpha/g)^{2\gamma}/g\Gamma(2\gamma+1)\exp(-\omega\alpha/g). \tag{30}$$

Due to the fact that we have chosen the same cutoff in G^0 and Q, we are able to obtain a closed expression for $D(\omega)$, which agrees with the low-frequency asymptote of Suzumura. It is easily seen that the sum rule quoted by Suzumura is satisfied by our result, i.e. $\alpha \int_0^\infty \mathrm{d}\omega[D(\omega) - D_0(\omega)] = 0$, where $D_0(\omega)$ is the non-interacting density of states.

3. Generalisation to spin-$\frac{1}{2}$ fermions

3.1. Introduction

The generalisation to spin-$\frac{1}{2}$ fermions is straightforward. One simply includes spinor indices $s = 1, 2$ in the electron fields. For H_0, we now have

$$H_0 = \sum_s \int \bar{\psi}_s(\sigma_3 p - p_f)\psi_s$$

and

$$H_1 = \int\int \tfrac{1}{2}(\bar{\psi}_{1\uparrow}\psi_{1\uparrow}, \bar{\psi}_{1\downarrow}\psi_{1\downarrow}, \bar{\psi}_{2\uparrow}\psi_{2\uparrow}, \bar{\psi}_{2\downarrow}\psi_{2\downarrow}) \begin{pmatrix} g_{4\parallel} & g_{4\perp} & g_{2\parallel} & g_{2\perp} \\ g_{4\perp} & g_{4\parallel} & g_{2\perp} & g_{2\parallel} \\ g_{2\parallel} & g_{2\perp} & g_{4\parallel} & g_{4\perp} \\ g_{2\perp} & g_{2\parallel} & g_{4\perp} & g_{4\parallel} \end{pmatrix} \begin{pmatrix} \bar{\psi}_{1\uparrow}\psi_{1\uparrow} \\ \bar{\psi}_{1\downarrow}\psi_{1\downarrow} \\ \bar{\psi}_{2\uparrow}\psi_{2\uparrow} \\ \bar{\psi}_{2\downarrow}\psi_{2\downarrow} \end{pmatrix} \tag{31}$$

where we have used the notation due to Sólyom and g are the bilocal functions of positions in general.

3.2. The partition function

As before, we introduce charge and spin density fields to decouple the four-fermion interaction term:

$$\begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ \sigma_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \bar{\psi}_{1\uparrow}\psi_{1\uparrow} \\ \bar{\psi}_{1\downarrow}\psi_{1\downarrow} \\ \bar{\psi}_{2\uparrow}\psi_{2\uparrow} \\ \bar{\psi}_{2\downarrow}\psi_{2\downarrow} \end{pmatrix}. \tag{32}$$

With these transformations, H_1 becomes

$$H_1 = \int\int (\rho_1\rho_2\sigma_1\sigma_2) \begin{pmatrix} v_1 & v_2 & & \\ v_2 & v_1 & & \\ & & \delta_1 & \delta_2 \\ & & \delta_2 & \delta_1 \end{pmatrix} \begin{pmatrix} \rho_1 \\ \rho_2 \\ \sigma_1 \\ \sigma_2 \end{pmatrix} \tag{33}$$

where

$$v_1 = \tfrac{1}{2}(g_{4\parallel} + g_{4\perp}) \qquad v_2 = \tfrac{1}{2}(g_{2\parallel} + g_{2\perp})$$
$$\delta_1 = \tfrac{1}{2}(g_{4\parallel} - g_{4\perp}) \qquad \delta_2 = \tfrac{1}{2}(g_{2\parallel} - g_{2\perp}).$$

Observe that the previous calculations for the spinless problem can be carried over for this case. Using the identities involving delta functionals and integrating out the electron fields and the auxiliary fields we obtain the partition function in terms of the spin and charge densities:

$$Z/Z_0 = \int \prod_a \mathrm{d}[\rho_a]\,\mathrm{d}[\sigma_a] \exp\left(-\int\int \zeta^*(p,\omega)K(p,\omega,v,\delta)\zeta(p,\omega)\right) \times\left[\int \prod_a \mathrm{d}[\rho_a]\,\mathrm{d}[\sigma_a] \exp\left(-\int\int \zeta^*(p,\omega)K(p,\omega,0,0)\zeta(p,\omega)\right)\right]^{-1} \tag{34}$$

where

$$\zeta^*(p,\omega) = (\rho_1\rho_2\sigma_1\sigma_2)_{-p,-\omega} \qquad \zeta(p,\omega) = \begin{pmatrix}\rho_1\\ \rho_2\\ \sigma_1\\ \sigma_2\end{pmatrix}_{p,\omega}$$

and

$$K(p,\omega,v,\delta) = \left(\begin{array}{cc:cc} (4A_1)^{-1}+v_1 & v_2 & & \\ v_2 & (4A_2)^{-1}+v_1 & & \\ \hdashline & & (4A_1)^{-1}+\delta_1 & \delta_2 \\ & & \delta_2 & (4A_2)^{-1}+\delta_1 \end{array}\right).$$

The block diagonal form of K shows that the spin fields and density fields do not interact with each other. This arises from the neglect of the back-scattering terms and results in undamped spin and charge density fluctuations. The expression for the free energy can be written down immediately since each block in K corresponds to the spinless problem:

$$Z/Z_0 = \exp\left(-\sum_p \ln[\sinh(\beta\varepsilon_\rho(p)/2)/\sinh(\beta|p|/2)][\sinh(\beta\varepsilon_\sigma(p)/2)/\sinh(\beta|p|/2)]\right)$$

and

$$F(T) = F_0(T) + \sum_{p>0}[\varepsilon_\rho(p)+\varepsilon_\sigma(p)-2p] + \sum_{p>0} 2T\ln\{[1-\exp(-\varepsilon_\rho(p)/T)][1-\exp(-\varepsilon_\sigma(p)/T)] \times[1-\exp(-p/T)]^{-2}\} \tag{35}$$

where

$$\varepsilon_\rho(p) = |p|[(1+v_1/\pi)^2-(v_2/\pi)^2]^{1/2}$$

and

$$\varepsilon_\sigma(p) = |p|[(1+\delta_1/\pi)^2-(\delta_2/\pi)^2]^{1/2}$$

are dispersion relations for spin and density excitations, describing doubly degenerate propagating modes. Again, for local interactions, up to a vacuum renormalisation energy, the free energy is quadratic in T for low T. The spin and charge fluctuations are equally important in contributing to the free energy.

3.3. The single-particle Green function at $T = 0$

The single-particle Green function was first calculated by Dzyaloshinsky and Larkin (1973), where the authors employed Ward identities in a diagrammatic analysis. The Green function was later calculated by Suzumura (1980) using the operator identities of Luther and Peschel. Since the spin-$\frac{1}{2}$ case can be split into two decoupled spinless problems, the procedure used from § 2.3 can be trivally adapted. Thus, the analogous expression for Q is now $Q_\rho + Q_\sigma$, with $Q_\rho(x, \tau)$ given by

$$-\pi \sum_{p\omega} p^{-1} \mathrm{e}^{-\alpha|p|}[1-\cos(px-\omega\tau)][(p+\mathrm{i}\omega)/(p^2+\omega^2)-(f_\rho p+\mathrm{i}\omega)/(u_\rho^2 p^2+\omega^2)]$$

where $f_\rho = 1 + v_1/\pi$, $u_\rho^2 = f_\rho^2 - (v_2/\pi)^2$ and Q_σ is obtained from Q_ρ by replacing ρ by σ and v by δ. The expression analogous to (A1.2) is $Q(x, t>0) = Q_\rho + Q_\sigma$ where

$$Q_\rho = \tfrac{1}{2}\ln\{[(x-u_\rho t+\mathrm{i}\alpha)(x+u_\rho t-\mathrm{i}\alpha)/\alpha^2]^{\gamma_\rho} \times (x-u_\rho t+\mathrm{i}\alpha)/(x-t+\mathrm{i}\alpha)\} \qquad \text{with} \qquad \gamma_\rho = [(f_\rho/u_\rho)-1]/4$$

and similarly for Q_σ, replacing ρ by σ.

Thus, $G_{1s}(x, t) = (2\pi)^{-1}\exp(\mathrm{i}p_\mathrm{f}x)\,W(x, t)$, where $W(x, t)$ is given by

$$\Theta(t)\{(x-u_\rho t+\mathrm{i}\alpha)^{-1/2}(x-u_\sigma t+\mathrm{i}\alpha)^{-1/2}[(x-u_\rho t+\mathrm{i}\alpha)(x+u_\rho t-\mathrm{i}\alpha)/\alpha^2]^{-\gamma_\rho} \times [(x-u_\sigma t+\mathrm{i}\alpha)(x+u_\sigma t-\mathrm{i}\alpha)/\alpha^2]^{-\gamma_\sigma}\} + \Theta(-t)\{\mathrm{cc}\}. \tag{36}$$

3.4. Momentum distribution at $T = 0$

We follow the steps taken in § 2.4. We can immediately write down the momentum distribution of the branch-1 electrons:

$$n_{1s}(p) = (2\pi\mathrm{i})^{-1}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{\mathrm{i}\alpha x(p_\mathrm{f}-p)}(x-\mathrm{i})^{-(\gamma_\sigma+\gamma_\rho+1)}(x+\mathrm{i})^{-(\gamma_\sigma+\gamma_\rho)}. \tag{37}$$

We find that, in the vicinity of p_f,

$$n_{1s}(p) - n_{1s}(p_\mathrm{f}) \sim -\tfrac{1}{2}\mathrm{sgn}(p-p_\mathrm{f})|(p-p_\mathrm{f})\alpha|^{\gamma_\sigma+\gamma_\rho}/\cos(\gamma_\sigma+\gamma_\rho) \qquad \text{for} \qquad 2(\gamma_\sigma+\gamma_\rho)<1$$
$$\sim |(p-p_\mathrm{f})\alpha| \qquad \text{for} \qquad 2(\gamma_\sigma+\gamma_\rho)>1$$

where

$$n_{1s}(p_\mathrm{f}) = \Gamma(\gamma_\sigma+\gamma_\rho+\tfrac{1}{2})/[2\sqrt{\pi}\Gamma(\gamma_\sigma+\gamma_\rho+1)]. \tag{38}$$

The above agrees with the results of Gutfreund and Schick (1968).

3.5. The density of states at $T = 0$

$$D_s(\omega) = -(2/\pi)\int_0^\infty \mathrm{d}t\,\cos(\omega t)\,\mathrm{Im}\left(\prod_{b=\rho,\sigma}(-u_b t+\mathrm{i}\alpha)^{-1/2}(1+\mathrm{i}u_b t/\alpha)^{-2\gamma_b}\right). \tag{39}$$

The explicit expression in this case is not known, since, from the Green function, the velocities of the two single-particle excitations are not equal to each other. However, its asymptotic form at $|\omega\alpha| \ll 1$ can be determined easily. We rescale the integrand such that $\cos(\omega t)$ is independent of ω. The leading term is then obtained by taking

the zero limit of $\omega\alpha$. Thus

$$D_s(\omega) = (2/\pi\omega\alpha)\int_0^\infty \mathrm{d}t \cos t \,\mathrm{Im}\left(\prod_{b=\rho,\sigma}(1+\mathrm{i}u_b t/\omega\alpha)^{-2\gamma_b-1/2}\right)$$
$$\sim (2/\pi\omega\alpha)\int_0^\infty \mathrm{d}t \cos t \,\mathrm{Im}\left(\prod_{b=\rho,\sigma}(\mathrm{i}u_b t/\omega\alpha)^{-2\gamma_b-1/2}\right)$$
$$= (\omega\alpha)^{2\gamma_\rho+2\gamma_\sigma} u_\rho^{-2\gamma_\rho-1/2} u_\sigma^{-2\gamma_\sigma-1/2} \sin 2\pi(\gamma_\rho+\gamma_\sigma)$$
$$\times \Gamma(-2\gamma_\rho-2\gamma_\sigma)/\pi \qquad \text{for} \qquad 2(\gamma_\rho+\gamma_\sigma)<1. \tag{40}$$

This agrees with the result of Suzumura (1980).

4. Conclusion

We have exploited the equivalence between a many-body problem in one spatial dimension and a $(1+1)$-dimensional field theory. The method of functional integration, which proves so powerful in the latter, is used to solve the Tomonaga–Luttinger model without large momentum transfer. We have checked our results for local interactions with previous work. For a long-range Coulomb interaction, the elementary excitations are found to be massive—a situation similar to the dynamic mass generation in $(\mathrm{QED})_2$. The advantage of our procedure lies in the introduction of auxiliary fields ϕ when the transformation to the density representation is performed. On integrating out these auxiliary fields, the original problem is bosonised in terms of the physical fluctuations of the system. On the other hand, one may integrate out the density fields instead, obtaining the effective action $\Lambda[\phi]$. The derivation of the Green function is then straightforward. Other n-point functions of the system can be calculated similarly. It is pleasing to see that the method of functional integration provides a clear and streamlined alternative to the conventional operator methods.

The next step in the development of our procedure is the inclusion of back scattering. The model is then equivalent to the massive Thirring model, the bosonisation of which has been carried out in the functional framework (see § 1). This leads to a sine-Gordon Hamiltonian (Heidenreich *et al* 1975) whose solution requires a renormalisation group analysis. We would like to point out that the kinetic energy in the Kondo–Anderson models can also be linearised. Perhaps the method proposed here might provide an alternative to the daunting methodology of the Bethe ansatz. At present, we are seeking to generalise the process of linearisation to higher dimensions. Envisaging possible relevance to the study of high-temperature superconductivity, Mattis (1987) has recently applied the bosonisation technique to the half-filled two-dimensional free-electron band with a square Fermi surface. However, apart from the obvious consequences of the nested Fermi surface, the instabilities of such a system remain to be investigated. We also hope to ascertain whether the smearing of the Fermi surface at zero temperature is a phenomenon with strong dimensional dependence or a pathology of the linear approximation to the free-fermion spectrum.

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Appendix 1. Calculation of $Q(x, t)$

In the imaginary-time formalism, it was found that $Q(x, \tau)$ is given by

$$\int_{-\infty}^{\infty} \mathrm{d}p(2\pi p)^{-1}\, \mathrm{e}^{-\alpha|p|} \int_{-\infty}^{\infty} \mathrm{d}\omega[1-\cos(px-\omega\tau)] \times[(fp+\mathrm{i}\omega)/(g^2p^2+\omega^2)-(p+\mathrm{i}\omega)/(p^2+\omega^2)]. \tag{A1.1}$$

We calculate the contribution from the first term in the second pair of square brackets, and then obtain the second term by setting $f=g=1$. Consider first the ω integration:

$$\begin{aligned}
&\int_{-\infty}^{\infty} \mathrm{d}\omega/(4\pi)\{2-\exp[\mathrm{i}(px-\omega\tau)]-\exp[-\mathrm{i}(px-\beta\tau)]\}(fp+\mathrm{i}\omega)/(g^2p^2+\omega^2) \\
&\quad = \int_{-\infty}^{\infty} \mathrm{d}\omega/(2\pi)[fp/(g^2p^2+\omega^2)] \\
&\qquad -(fp-\partial_\tau)\int_{-\infty}^{\infty} \mathrm{d}\omega/(4\pi)\exp[\mathrm{i}(px-\omega\tau)]/(g^2p^2+\omega^2) \\
&\qquad -(fp+\partial_\tau)\int_{-\infty}^{\infty} \mathrm{d}\omega/(4\pi)\exp[-\mathrm{i}(px-\omega\tau)]/(g^2p^2+\omega^2) \\
&\quad = (f/g)\exp(-g|p\tau|)-\tfrac{1}{4}\exp(\mathrm{i}px-g|p\tau|)[(f/g)\operatorname{sgn}p+\operatorname{sgn}\tau] \\
&\qquad -\tfrac{1}{4}\exp(-\mathrm{i}px-g|p\tau|)[(f/g)\operatorname{sgn}p-\operatorname{sgn}\tau].
\end{aligned}$$

Including the $f=g=1$ term, we get

$$\begin{aligned}
Q(x,\tau) = &\int_{-\infty}^{\infty} \mathrm{d}p/(2p)\exp(\mathrm{i}px-\alpha p)\{\exp(-p\tau)(1+\operatorname{sgn}\tau) \\
&-\exp(-gp\tau)[(f/g)\operatorname{sgn}p+\operatorname{sgn}\tau]\} \\
&+\int_{0}^{\infty} \mathrm{d}p/(2p)\exp(-\mathrm{i}px-\alpha p)\{\exp(-p\tau)(1-\operatorname{sgn}\tau) \\
&-\exp(-gp\tau)[(f/g)\operatorname{sgn}p-\operatorname{sgn}\tau]\} \\
&+\int_{0}^{\infty} \mathrm{d}p/p\exp(-\alpha|p|)[(f/g)-1].
\end{aligned}$$

(We have changed the integration limits from $(-\infty, \infty)$ to $(0, \infty)$.) Now we perform the Wick rotation: $\tau \to \mathrm{i}t$, $\operatorname{sgn}\tau \to \operatorname{sgn}t$, $Q(x,\tau) \to Q(x,t)$. This gives

$$\begin{aligned}
Q(x,t) = &\int_{0}^{\infty} \mathrm{d}p/(2p)\{\exp[\mathrm{i}p(x-t+\mathrm{i}\alpha)](1+\operatorname{sgn}t) \\
&-\exp[\mathrm{i}p(x-gt+\mathrm{i}\alpha)][(f/g)\operatorname{sgn}p+\operatorname{sgn}t]\} \\
&+\int_{0}^{\infty} \mathrm{d}p/(2p)\{\exp[\mathrm{i}p(x+t+\mathrm{i}\alpha)](1-\operatorname{sgn}t) \\
&-\exp[-\mathrm{i}p(x+gt+\mathrm{i}\alpha)][(f/g)\operatorname{sgn}p-\operatorname{sgn}t]\} \\
&+\int_{0}^{\infty} \mathrm{d}p/p\exp(-\alpha p)[(f/g)-1].
\end{aligned}$$

Note that changing the sign of t is equivalent to a complex conjugation: $Q(x, t<0) = Q(x, |t|)^*$. It suffices to consider $t>0$. After some rearrangement, $Q(x, t)$ becomes

$$\int_0^\infty \mathrm{d}p\, p^{-1} \exp(-\alpha p)[[\exp[\mathrm{i}p(x-t)] - \exp[\mathrm{i}p(x-gt)] + \gamma\{2 - \exp[\mathrm{i}p(x-gt)] - \exp[\mathrm{i}p(x+gt)]\}]]$$

where $\gamma = 2(f/g - 1)$.

$\partial_x Q(x, t)$ is integrated to give

$$Q(x, t) = \ln\{[(x-gt+\mathrm{i}\alpha)(x+gt-\mathrm{i}\alpha)/\alpha^2]^\gamma (x-gt+\mathrm{i}\alpha)/(x-t+\mathrm{i}\alpha)\} \tag{A1.2}$$

where the integration constant is chosen such that $Q(x=0, t=0) = 0$.

Appendix 2. Momentum distribution

Let $q = |(p - p_f)\alpha|$ and consider first $p > p_f$.

$$n_1(q) = (2\pi\mathrm{i})^{-1} \int_{-\infty}^{\infty} \mathrm{d}x \exp(-\mathrm{i}qx)(x-\mathrm{i})^{-\gamma-1}(x+\mathrm{i})^{-\gamma}$$

$$= (2\pi)^{-1}[\Gamma(\gamma+1)\Gamma(\gamma)]^{-1} \int_0^\infty \mathrm{d}a \int_0^\infty \mathrm{d}b\, a^{\gamma-1} b^\gamma \times \exp[-(a+b)] \int_{-\infty}^{\infty} \mathrm{d}x \exp[\mathrm{i}(a-b-q)x] \tag{A2.1}$$

where we have used the identity

$$(A \pm \mathrm{i}B)^{-\alpha} = (\mp\mathrm{i})^\alpha [\Gamma(\alpha)]^{-1} \int_0^\infty \mathrm{d}x \exp[\mathrm{i}x(\pm A + \mathrm{i}B)] x^{\alpha-1}$$

to arrive at the last equation. The x integration gives $2\pi\delta(a-b-q)$. Since $b+q>0$ for the range of integration, we have

$$n_1(p > p_f) = q^{2\gamma}\, \mathrm{e}^{-q} \Psi(\gamma+1, 2\gamma+1; 2q)/\Gamma(\gamma) \tag{A2.2}$$

where

$$\Psi(\alpha, \beta; z) = (1/\Gamma(\alpha)) \int_0^\infty \mathrm{d}t\, \mathrm{e}^{-zt} t^{\alpha-1} (t+1)^{\beta-\alpha-1} \qquad (\mathrm{Re}\ \alpha > 0).$$

For $p < p_f$, we obtain similarly

$$n_1(p < p_f) = q^{2\gamma}\, \mathrm{e}^{-q} [\Gamma(\gamma+1)]^{-1} \Psi(\gamma+1, 2\gamma+1; 2q). \tag{A2.3}$$

Combining (A2.2) and (A2.3) gives the desired result, equation (27).

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